The Journal of Fourier Analysis and Applications

Volume 13, Issue 6, 2007

Image Projection and Representation on $\mathbb{S}^n$

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Communicated by Stephan Dahlke

ABSTRACT. In signal processing, communications, and other branches of information technologies, it is often desirable to map the higher-dimensional signals on $\mathbb{S}^n$. In this article we introduce a novel method of representing signals on $\mathbb{S}^n$. This approach is based on geometric function theory, in particular on the theory of quasiregular mappings. The importance of sampling is underlined, and new geometric sampling theorems for general manifolds are presented.

1. Introduction

One of the major applications of harmonic analysis and in particular Fourier analysis, is signal processing. In recent years it became common amongst the signal processing community, to consider signals as Riemannian manifolds embedded in higher-dimensional spaces. Usually, the embedding manifold is taken to be $\mathbb{R}^n$ yet, other possibilities are also considered.

For instance, in [26] images are considered as surfaces embedded in higher-dimensional manifolds, where a gray scale image is a surface in $\mathbb{R}^3$, and a color image is a surface embedded in $\mathbb{R}^5$, each color channel representing a coordinate of the ambient space. In both cases the intensity, either gray scale or color, is considered as a function of the two spatial coordinates (x, y) and thus the surface may be equipped with a metric induced by this function. The question of smoothness of the function is in general omitted, if numerical schemes are used for the approximations of derivatives, whenever this is necessary.

A major advantage of such a viewpoint of signals is the ability to apply mathematical tools traditionally used in the study of Riemannian manifolds, for signal processing as well. For example, in medical imaging it is often convenient to treat CT/MRI scans, as Riemannian surfaces in $\mathbb{R}^3$. One can then borrow techniques from differential geometry and geometric analysis, such as quasiconformal/quasiisometric maps between Riemannian manifolds, in

Math Subject Classifications. Primary: 68U10, 30C65, 57R05; secondary: 57R10, 57R12, 53C21.
Keywords and Phrases. Image representation, geometric sampling, quasiregular mapping, Alexander's method, spherical metric.

ISSN 1069-5869 DOI: 10.1007/s00041-006-6906-z

order to flatten the curved scanned surface. It is then possible to get a planar representation of the surface while keeping geometric features, essential for medical analysis, as minimally distorted as possible (see [12, 4], and [23]). Since surfaces such as that of the cortex are highly curved and folded, it is inevitable that some distortion of angles, length and area will occur while surfaces are flattened. However, by using quasiconformal flattening methods one may keep these distortions controlled. For brain scans, it is natural to consider maps of the cortical surface onto the 2-sphere, $\mathbb{S}^2$ [13]. For higher-dimensional signals one can consider quasiregular maps onto $\mathbb{S}^n$.

We address this problem in detail in Section 5, where we present three alternative methods for representing signals on $\mathbb{S}^n$. One of the methods is based on classical results of Gehring and Väisälä, [9], already applied by the authors in flattening of curved surfaces onto a planar region, (see [4] and [23]). Its application for spherical representation of images (and more general signals) is considered here for the first time. The other two versions are presented here for the first time, where one of them is based on the use of the chordal metric on $\mathbb{S}^n$, and the other (developed in [22]) represents a geometric extension of a classical result of J. Alexander, regarding the construction of branched covering maps from a Riemannian manifold to $\mathbb{S}^n$ [1]. Some of the results employed here are also announced for the first time. Also, while conformal and quasiconformal mappings have been extensively employed in image processing, considering their higher-dimensional generalizations is novel, as is the introduction herein, for the first time, of their branched counterparts (i.e., quasiregular mappings) for the representation of folded tissues and textures with singularities.

In Section 2, we briefly present an overview of some essential definitions and notations on quasiregular maps, and relations between quasiconformal and quasiisometric maps. The former are used to control angle dilatation, while the latter are used for controlling the distortion of length and area.

At the heart of the geometric methods presented in this article for signal processing, lies the assumption that the signal under consideration is well sampled. Sampling theory is fundamental to studies in which Fourier analysis is employed. However, concrete sampling theorems, based on this theory, are valid essentially for one-dimensional signals (see [16, 29], and [27]).

In Section 3 we present results relevant to the existence of fat triangulations of manifolds and to their relation to sampling, reproducing and quasiconformal mapping of Riemannian manifolds. The main results employed here are those of [22]. We also review the problem of smoothing of manifolds. The smoothing method adopted here is the one developed in [18]. Subsequently, in Section 4, we present geometrical sampling theorems for signals given as Riemannian manifolds. This is done for both smooth and nonsmooth signals. Though geometric sampling of manifolds is not a new idea and previous examples may be found in [2] and [17], such sampling theorems are presented in a full rigorous manner for the first time and, moreover, they apply also for higher-dimensional manifolds, while the examples mentioned above are valid merely for two-dimensional manifolds, i.e., for surfaces. Moreover, unlike in previous works, sampling theorems for nonsmooth signals are also obtained.

2. Theoretical Background

2.1 Quasiconformal Mappings

Definition 1. Let $D \subseteq \mathbb{R}^n$ be a domain; $n \geq 2$ and let $f : D \to \mathbb{R}^n$ be a continuous mapping. f is called

(1) *quasiregular* iff (i) f belongs to $W^{1,n}_{\text{loc}}(D)$ and (ii) there exists $K \geq 1$ such that:

$$\left|f'(x)\right|^n \leq K J_f(x) \ a.e.\,, \tag{2.1}$$

where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|=1}|f'(x)h|$, and where $J_f(x) = \det f'(x)$.

(2) *quasiconformal* iff $f : D \to f(D)$ is a quasiregular homeomorphism.

(3) *quasimeromorphic* iff $f : D \to \widehat{\mathbb{R}^n}$, $\widehat{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ can be verified by conjugation with auxiliary Möbius transformations.

The smallest number K that satisfies (2.1) is called the *outer dilatation*, $K_O(f)$, of f.

[Here $W^{1,n}_{\text{loc}}(D)$ denotes the *Sobolev space* (see [20], pp. 5–11).]

One can extend the above definitions to oriented, connected $\mathcal{C}^\infty$ Riemannian manifolds as follows.

Definition 2. Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. f is called *locally quasiregular* iff for every $x \in M^n$, there exist coordinate charts (U_x, φ_x) and $(V_{f(x)}, \psi_{f(x)})$, such that $f(U_x) \subseteq V_{f(x)}$ and $g = \psi_{f(x)} \circ f \circ \varphi_x^{-1}$ is quasiregular.

If f is locally quasiregular, then $T_x f : T_x(M^n) \to T_{f(x)}N^n$ exists for a.e. $x \in M^n$.

Definition 3. Let M^n, N^n be oriented, connected $\mathcal{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f : M^n \to N^n$ be a continuous function. f is called *quasiregular* iff

(1) f is locally quasiregular and

(2) there exists K, $1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x)\,, \tag{2.2}$$

for a.e. $x \in M^n$.

If f is quasiregular, then there exists $K \geq 1$ such that the following inequality holds a.e. in M^n:

$$J_f(x) \leq K' \cdot \inf_{|h|=1} |T_x f h|^n\,. \tag{2.3}$$

By analogy with the outer dilatation we have the following definition.

Definition 4. The smallest number K' that satisfies inequality (2.3) is called the inner dilatation $K_I(f)$ of f. The maximal dilatation of f is defined as $\max(K_O(f), K_I(f))$. A mapping f is called K-quasiregular iff there exists $K \geq 1$ such that $K(f) \leq K$.

The dilatations are simultaneously finite or infinite. Indeed, the following inequalities hold:

$$K_I(f) \leq K_O^{n-1}(f) \text{ and } K_O(f) \leq K_I^{n-1}(f)\,.$$

Definition 5. A continuous map $f : X \to Y$ between topological spaces is called:

(1) *open* iff it maps open sets onto open sets;
(2) *discrete* iff $f^{-1}(y)$ is discrete (i.e., it consists of isolated points), for any y in Y;
(3) *a branched cover* iff it is discrete and open.

If $f : D \to \mathbb{R}^n$ is a nonconstant quasiregular mapping, then f is an orientation preserving branched cover (see, e.g., [20], VI. 5.7.). This is a direct consequence of the classical result of Reshetnyak (see, e.g., [20]).

Definition 6. Let $f : M^n \to N^n$ be a quasiregular mapping. The set $B_f = \{x \in M^n \mid f$ is not a local homeomorphism at $x\}$ is called the *singular set* and $f\big(B_f\big)$ is called the *branch set* of f.

2.2 Quasiisometries and Admissible Hypersurfaces

Definition 7. Let $D \subset \mathbb{R}^n$ be a domain. A homeomorphism $f : D \to \mathbb{R}^n$ is called a *quasi-isometry* (or a *bi-lipschitz mapping*), if there exists $1 \leq C < \infty$ such that

$$\frac{1}{C}|p_1 - p_2| \leq |f(p_1) - f(p_2)| \leq C|p_1 - p_2|\,, \text{ for all } \quad p_1, p_2 \in D\,, \tag{2.4}$$

where "$|\cdot|$" denotes the standard (Euclidean) metric on $\mathbb{R}^n$. The number $C(f) = \min\{C \mid f$ is a quasiisometry$\}$ is called the minimal distortion of f (in D).

Remark 1. Evidently, the above definition readily extends to general metric spaces. In particular, for the case of hypersurface embedded in $\mathbb{R}^n$, distances are the induced intrinsic distances on the surface.

If f is a quasiisometry, then

$$K(f) \leq C(f)^{2(n-1)}\,. \tag{2.5}$$

It follows that any quasiisometry is a quasiconformal mapping (while, evidently, not every quasiconformal mapping is a quasiisometry).

Definition 8. Let $S \subset \mathbb{R}^n$ be a connected, $(n-1)$-dimensional set. S is called *admissible* (or an *admissible hypersurface*) iff for any $p \in S$, there exists a quasiisometry i_p such that for any $\varepsilon > 0$ there exists a neighborhood $U_p \subset \mathbb{R}^n$ of p, such that $i_p : U_p \to \mathbb{R}^n$ and $i_p(S \cap U_p) = D_p \subset \mathbb{R}^{n-1}$, where D_p is a domain, and such that $C(i_p)$ satisfies:

$$\text{(i) } \sup_{p \in S} C(i_p) < \infty\,, \text{ and (ii) } \operatorname{ess\,sup}_{p \in S} C(i_p) \leq 1 + \varepsilon\,.$$

2.3 The Projection Map

Let $S \subset \mathbb{R}^n$ be a set homeomorphic to an $(n-1)$-dimensional domain, $\vec{n}$ be a fixed unitary vector, and $p \in S$, such that there exists a neighborhood $V \subset S$, such that $V \simeq D^{n-1}$, where $D^{n-1} = \{x \in \mathbb{R}^{n-1} \mid ||x|| \leq 1\}$. Moreover, suppose that for any $q_1, q_2 \in S$, the acute angle $\measuredangle(q_1q_2, \vec{n}) \geq \alpha$. We refer to the last condition as *the Geometric Condition* or *Gehring Condition* (see [9]). Then, for any $x \in V$ there exists a unique representation of the following form:

$$x = q_x + u\vec{n}\,,$$

where q_x lies on the hyperplane through p, which is orthogonal to $\vec{n}$ and $u \in \mathbb{R}$. Define:

$$Pr(x) = q_x\,.$$

Remark 2. $\vec{n}$ need not be the normal vector to S at p.

By [7], Corollary 1, p. 338, we have that for any $p_1, p_2 \in S$ and any $a \in \mathbb{R}_+$ the following inequalities hold:

$$\frac{a}{A}\,|p_1 - p_2| \le |Pr(p_1) - Pr(p_2)| \le A|p_1 - p_2|\,,$$

where

$$A = \frac{1}{2}\big[(a\csc\alpha)^2 + 2a + 1\big]^2 + \frac{1}{2}\big[(a\csc\alpha)^2 - 2a + 1\big]^2\,.$$

In particular, for $a = 1$ we get that

$$C(f) \le \cot\alpha + 1 \tag{2.6}$$

and

$$K(f) \le \left(\left(\frac{1}{2}(\cot\alpha)^2 + 4\right)^{\frac{1}{2}} + \frac{1}{2}\cot\alpha\right)^n \le (\cot\alpha + 1)^n\,. \tag{2.7}$$

From the above discussion we conclude that $S \subset \mathbb{R}^3$ is an admissible hypersurface if for any $p \in S$ there exists $\vec{n}_p$ such that for any $\varepsilon > 0$ there exists $U_p \simeq D^2$ such that for any $q_1, q_2 \in U_p$ the acute angle $\measuredangle(q_1q_2, \vec{n}_p) \ge \alpha$, where

$$\text{(i)}\ \inf_{p\in S}\alpha_p > 0\,,\ \text{and (ii) ess}\inf_{p\in S}\alpha_p \ge \frac{\pi}{2} - \varepsilon\,.$$

Example 1. Any hypersurface in $S \in \mathbb{R}^3$, that admits a well-defined continuous turning tangent plane at any point $p \in S$ is admissible.

For application of these notions to Medical Imaging, see [4, 23].

3. Triangulation and Sampling

We first recall a few classical definitions and notations (for further details see [18]).

Definition 9. Let $a_0, \dots, a_m \in \mathbb{R}^n$. $\{a_i\}_{i=1}^m$ are said to be *independent* iff the vectors $v_i = a_i - a_0$, $i = 1, \dots, m$; are linearly independent.

Let $a_0, \dots, a_m \in \mathbb{R}^n$ be independent points. The set $\sigma = a_0a_1\dots a_m = \{x = \sum\alpha_i a_i \,|\, \alpha_i \ge 0, \sum\alpha_i = 1\}$ is called the *m-simplex spanned by* $a_0, \dots, a_m$. The points $a_0, \dots, a_m$ are called the *vertices* of σ.

The numbers α_i are called the *barycentric coordinates* of σ. The point $\tilde{\sigma} = \frac{1}{m+1}\sum\alpha_i$ is called the *barycenter* of σ.

If $\{a_0, \dots, a_k\} \subseteq \{a_0, \dots, a_m\}$, then $\tau = a_0\dots a_k$ is called a *face* of σ, and we write $\tau < \sigma$.

Definition 10. A collection K of simplices is called a *simplicial complex* if

(1) if $\tau < \sigma$, then $\tau \in K$.
(2) Let $\sigma_1, \sigma_2 \in K$ and let $\tau = \sigma_1 \cap \sigma_2$. Then $\tau < \sigma_1, \tau < \sigma_2$.
(3) K is locally finite.

$|K| = \bigcup_{\sigma \in K} \sigma$ is called the *underlying polyhedron* (or *polytope*) of K.

Definition 11. A complex K' is called a *subdivision* of K iff

(1) $K' \subset K$;
(2) if $\tau \in K'$, then there exists $\sigma \in K$ such that $\tau \subseteq \sigma$.

Let K be a simplicial complex and let $L \subset K$. If L is a simplicial complex, then it is called a *subcomplex* of K.

Definition 12. Let K be a *simplicial complex*, let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}_+^*$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a *δ-approximation to* f iff:

(i) There exists a *subdivision* K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;
(ii) $d_{\text{eucl}}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{\text{eucl}}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{\text{eucl}}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.
(Here and below $|K|$ denotes *the underlying poyhedron* of K.)

Definition 13. Let K' be a subdivision of K, $U = \overset{\circ}{U} \subset |K|$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a *$\delta$-approximation* of f on U, iff conditions (ii) and (iii) of Definition 6. hold for any $a \in U$.

The most natural and intuitive δ-approximation to a given mapping f is the *secant map induced by f*.

Definition 14. Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \to \mathbb{R}^n$, defined by $L_s(v) = f(v)$, where v is a vertex of s is called the *secant map induced by f*.

3.1 Triangulations

Fat Triangulations

We now proceed to show that the apparent "naive" secant approximation of surfaces (and higher-dimensional manifolds) represents a good approximation, both in distances and in angles, provided that the secant approximation induced by a triangulation that satisfies a certain nondegeneracy condition called "fatness" (or "thickness").

Definition 15. A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is *φ-fat* if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$; where r denotes the radius of the inscribed sphere of τ (*inradius*) and R denotes the radius of the circumscribed sphere of τ (*circumradius*). A triangulation of a submanifold of $\mathbb{R}^n$, $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is *φ-fat* if there exists $\varphi \geq 0$ such that all its simplices are *φ-fat*.

Proposition 1 ([8]). *Let τ be a k-dimensional simplex, and let $\sigma < \tau$. Denote by φ the fatness of τ and by $\measuredangle(\tau, \sigma)$ the (internal) dihedral angle of the face $\sigma < \tau$. Also, let* $\mathrm{Vol}_j(\sigma)$ *and* $\mathrm{diam}\,\sigma$ *stand for the Euclidian j-volume and the diameter of σ, respectively.*

(If $\dim \sigma = 0$*, then* $\mathrm{Vol}_j(\sigma) = 1$*, by convention.) Then there exists a constant* $c(k)$ *that depends solely upon the dimension* k *of* τ *such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma < \tau} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{3.1}$$

and

$$\varphi(\tau) \leq \frac{\mathrm{Vol}_j(\sigma)}{\mathrm{diam}^j\,\sigma} \leq c(k) \cdot \varphi(\tau)\,. \tag{3.2}$$

Condition (3.1) is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition (3.2) expresses fatness as given by "large area/diameter." Diameter is relevant, since fatness is independent of scale.

The importance of fatness of triangulations, for attaining good approximations, is underlined in the following proposition that represents the desired approximation result for the secant map.

Proposition 2 ([18], Lemma 9.3). *Let* σ *be a simplex and let* $f : \sigma \to \mathbb{R}^n$ *be of class* $\mathcal{C}^k$*. Then, for any* $\delta, \varphi_0 > 0$*, there exists* $\varepsilon > 0$*, such that, for any* $\tau < \sigma$*, satisfying the conditions:* (i) $\mathrm{diam}(\tau) < \varepsilon$ *and* (ii) $\varphi(\tau) > \varphi_0$*, the secant map* L_τ *is a* δ*-approximation of* $f|\tau$*.*

Proof. Suffice to prove the following two assertions: (i) $F_b(x) = f(b) + Df(b)\cdot(x - b)$, where b denotes the barycenter of σ, is a $\delta/2$-approximation to f on a sufficient small neigbourhood of b, and (ii) if $\tau < \sigma$ satisfies the conditions from the statement of the theorem, then L_σ is a $\delta/2$-approximation to F_b.

Proof of (i)**.** Follows immediately from the definition of Df. We impose the additional requirement $||f(x) - F_b(x)||/||x - b|| < \delta\varphi_0/4$, for $||x - b|| < \varepsilon$. (Here $||\cdot||$ denotes the Euclidean norm.)

Before we proceed further we need the following result: Let $L, F : \tau \to \mathbb{R}^n$ be linear maps, such that $||L(x) - F(x)|| < c$, for all $x \in \tau$. Then, it results immediately from (i) that $||DL(x)\cdot\mathbf{u} - DF(x)\cdot\mathbf{u}|| \leq c/r(\tau)$, for all $\mathbf{u}$ in the plane of τ, $||\mathbf{u}|| = 1$.

Proof of (ii)**.** Let $v_0, \ldots, v_k$ be the vertices of τ, and let $x \in \tau, x = \sum \alpha_i v_i$. Then, by the linearity of L_s and F_s it follows that $L_s(x) = \sum \alpha_i L_s(v_i) = \sum \alpha_i f(v_i)$ and $F_b(x) = \sum \alpha_i F_b(v_i)$. Hence:

$$||L_s(x) - F_b(x)|| = \left\| \sum \alpha_i \frac{||f(x) - F_b(x)||}{||x - b||} \right\| \leq \max ||f(x) - F_b(x)||\,,$$

but $||f(x) - F_b(x)|| < \delta/2$ and $||L_s(x) - F_b(x)|| < \delta/2$, for all $||x - b|| < \varepsilon$. Moreover, $||f(x) - F_b(x)||/||x - b|| < \delta\varphi_0/4$, for all $||x - b|| < \varepsilon$, and, since $\varphi_0 \leq r(\tau)/\mathrm{diam}(\tau)$, it follows that:

$$||L_s(x) - F_b(x)|| < \max ||v_i - b||\delta\varphi_0/4 \leq \mathrm{diam}(\tau)\delta\varphi_0/4 \leq \delta\, r(\tau)/4\,.$$

This concludes the proof of (ii), and, hence, of the proposition. □

The main existence result of fat triangulations to which we make appeal herein is the following theorem.

Theorem 1 ([21]). *Let* M^n *be an n-dimensional* $\mathcal{C}^1$ *Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of* ∂M^n *can be extended to a fat triangulation of* M^n*.*

Corollary 1. *Let M^n be a manifold as in Theorem 1 above. Then M^n admits a fat triangulation.*

Remark 3. Theorem 1 above holds, in fact, even without the finiteness and compactness conditions imposed on the boundary components. Moreover, in low dimensions one can also discard the smoothness condition (see [22]).

Smoothing of Manifolds

In this section we focus our attention on the problem of smoothing of manifolds. That is, approximating a manifold of differentiability class $\mathcal{C}^r$, $r \geq 1$, by manifolds of class $\mathcal{C}^\infty$. Of special interest is the case where $r = 0$, which we defer for later study. Later, when posing our sampling theorem we will make a use of this in two respects. One of them will be as a post-processing step where, after reproducing a PL manifold out of the samples, we can smoothen it to get a smooth reproduced manifold. Another aspect in which smoothing is useful is as a pre-processing step, when we wish to extend the sampling theorem to manifolds which are not necessarily smooth. Smoothing will take place followed by sampling of the smoothed manifold, yielding a sampling for the nonsmooth one as well. As a major reference to this we use [18], Chapter 4.

Smoothing will be done while using a smoothing convolution kernel. This kernel will be a $\mathcal{C}^\infty$ function. Before introducing this kernel we present the very basic idea of partition of unity. At a first glance it seems unrelated, but in fact this is the core of the smoothing process.

Lemma 1. *For every $0 < \epsilon < 1$ there exists a $\mathcal{C}^\infty$ function $\psi_1 : \mathbb{R} \to [0,\ 1]$ such that, $\psi_1 \equiv 0$ for $|x| \geq 1$ and $\psi_1 = 1$ for $|x| \leq (1-\epsilon)$. Such a function is called partition of unity.*

Let $c^n(\epsilon)$ be the ϵ cube around the origin in $\mathbb{R}^n$ (i.e., $X \in \mathbb{R}^n$; $-\epsilon \leq x_i \leq \epsilon\,, \forall i = 1,\dots,n$). We can use the above partition of unity in order to obtain a non-negative $\mathcal{C}^\infty$ function, ψ, on $\mathbb{R}^n$ such that $\psi = 1$ on $c^n(\epsilon)$ and $\psi \equiv 0$ outside $c^n(1)$. Define $\psi(x_1,\dots,x_n) = \psi_1(x_1)\cdot\psi_1(x_2)\cdots\psi_1(x_n)$.

We now introduce the main theorem regarding smoothing of PL-manifolds.

Theorem 2 ([18]). *Let M be a $\mathcal{C}^r$ manifold, $1 \leq r \leq \infty$, and $f_0 : M \to \mathbb{R}^k$ a $\mathcal{C}^r$ embedding. Then, there exists a $\mathcal{C}^\infty$ embedding $f_1 : M \to \mathbb{R}^k$ which is a δ-approximation of f_0.*

The above theorem is a consequence of the following lemma concerning smoothing of maps.

Lemma 2 ([18]). *Let U be an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$. Let A be a compact subset of an open set V such that $\overline{V} \subset U$. Let $f_0 : U \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, $r \geq 1$. Let δ be a positive number. Then there exists a map $f_1 : U \to \mathbb{R}^n$ such that*

(1) *f_1 is $\mathcal{C}^\infty$ on A.*
(2) *$f_1 = f_0$ outside V.*
(3) *f_1 is a δ-approximation of f_0*
(4) *f_1 is $\mathcal{C}^r$-homotopic to f_0 via a homotopy f_t satisfying (2) and (3) above.*

Proof. We may assume that $\overline{V}$ is compact and that U is open in $\mathbb{R}^n$, (if U is open $\mathbb{H}^n$ we can extend f_0 in a neighborhood of the boundary $\partial\mathbb{H}^n$). Let W be an open set containing

A such that $\overline{W} \subset V$. Let $\psi : \mathbb{R}^m \to \mathbb{R}^+$ be a $\mathcal{C}^\infty$ map, such that $\psi = 1$ on A and $\psi \equiv 0$ outside W.

Define $g = \psi \cdot f$, then $g : \mathbb{R}^m \to \mathbb{R}^n$ satisfying $g = f$ on A and $g \equiv 0$ outside W. Inside A, g is of the same differentiability type as f whereas outside W it is $\mathcal{C}^\infty$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is positive on $\mathrm{int}(c^m(\epsilon))$ and vanishes outside $\overline{c^m(\epsilon)}$. Here ϵ is some positive number yet to be defined. Further assume that $\int_{\mathbb{R}^m} \varphi = 1$. Such a function can be obtained by, say, taking a partition of unity, φ_0, supported on some open subset of $c^m(\epsilon)$ and then factorizing it by $\int_{\mathbb{R}^m} \varphi_0$. Such a function will be called a convolution kernel.

For $x \in \mathbb{R}^m$, define

$$h(x) = \int_{c^m(\epsilon)} \varphi(y) g(x+y)\, dy\,. \tag{3.3}$$

Choose ϵ so that $\sqrt{m}\epsilon < d(W, \mathcal{R}^m \backslash V)$. then $h \equiv 0$ outside V. Let

$$f_1(x) = f_0(x) \cdot (1 - \psi(x)) + h(x)\,.$$

Since ψ and h vanishes outside V conclusion (2) of the lemma is fulfilled.

Inside A we have $f_1(x) = h(x)$. Since

$$h = \int_{c^m(\epsilon)} \varphi(y) g(x+y)\, dy = \int_{W + c^m(\epsilon)} \varphi(z-x) g(z)\, dz = \int_{\mathbb{R}^m} \varphi(z-x) g(z)\, dz\,,$$

and φ is $\mathcal{C}^\infty$, h is also $\mathcal{C}^\infty$ inside W and in particular on A, thus fulfilling conclusion (1).

By its definition $f_1 = f_0 + (h - g)$, so we have to choose ϵ small enough so that h is a δ-approximation to g. By the mean value theorem (h is actually some weighted mean of g), we have:

$$h^i(x) = g^i\big(x + y^i\big)\,,$$
$$\frac{\partial h^i}{\partial x^j} = \frac{\partial g^i\big(x + y^{ij}\big)}{\partial x^j}\,,$$

where y^i and y^{ij} are points in $c^m(\epsilon)$. We only have to take care that ϵ is so small that

$$\big|g^i(x) - g^i\big(x'\big)\big| < \delta\,,$$

and

$$\Big|\frac{\partial g^i}{\partial x^j}(x) - \frac{\partial g^i}{\partial x^j}\big(x'\big)\Big| < \delta\,,$$

for

$$\big|x - x'\big| < \epsilon\,.$$

This completes the proof of part (3).

Finally, we construct the desired homotopy between f_0 and f_1. Let $\alpha(t)$ be a monotonic $\mathcal{C}^\infty$ function such that, $\alpha = 0$ for $0 \le t \le 1/3$ and $\alpha = 1$ for $2/3 \le t \le 1$.

Put

$$f_t(x) = \alpha(t) f_1(x) + (1 - \alpha(t)) f_0(x)\,. \tag{3.4}$$

Then, outside V $f_t \equiv f_0$ and f_t is a $\mathcal{C}^r$ homotopy between f_0 and f_1 satisfying,

$$|f_t - f_0| < \delta\,,$$

and whenever f_0 is differentiable,

$$|f_t - f_0| < \delta .$$

This completes the proof. □

Remark 4. In the proof of the isometric embedding theorem, J. Nash [19] used a modified version of the smoothing process presented herein. Nash's idea was to define a radially symmetric convolution kernel φ, by taking its Fourier transform, $\hat{\varphi}$, to be a radially symmetric partition of unity. In so doing one can use a scaling process where for each N define the smoothing operator of g to be

$$h_N g(x) = \int_{\mathbb{R}^m} \varphi(z) g(x + z/N)\, dz = \int_{\mathbb{R}^m} \varphi_N(z - x) g(z)\, dz ,$$

where $\varphi_N(z) = N^m \varphi(Nz)$. So we have that the Fourier transform of φ_N satisfies

$$\hat{\varphi}_N(\omega) = \hat{\varphi}(\omega/N) .$$

Note, that from this we have more smoothing for small N (the partition of unity is taken over a larger neighborhood) while for large N we have less smoothing resulting with a better approximation. In that case the approximation is faithful not only to the signal and its first derivative as in the classical approach, but also to higher order derivatives, if existent, of course.

4. Sampling Theorems

We employ results regarding the existence of fat triangulations, to prove sampling theorems for Riemannian manifolds embedded in some Euclidean space. (For a detailed discussion of sampling theorems see [24].)

Theorem 3. *Let $\Sigma^n \subset \mathbb{R}^{n+1}, n \geq 2$ be a connected, not necessarily compact, smooth hypersurface, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\left(\frac{1}{k(p)}\right)$, where $k(p) = \max\{|k_1|, ..., |k_n|\}$, and where $k_1, ..., k_n$ are the principal curvatures of Σ^n, at the point $p \in \Sigma^n$.*

Proof.

The existence of the sampling scheme follows immediately from Corollary 1, where the sampling points(points of the sampling) are the vertices of the triangulation. The fact that the density is a function solely of $k = max\{|k_1|, ..., |k_n|\}$ follows from the proof of Theorem 3.11 (see [22], [24]) and from the fact that the osculatory radius $\omega_\gamma(p)$ at a point p of a curve γ equals $1/k_\gamma(p)$, where $k_\gamma(p)$ is the curvature of γ at p ; hence the maximal osculatory radius (of Σ) at p is: $\omega(p) = \max\{|k_1|, ..., |k_n|\} = \max\{\frac{1}{\omega_1}, ..., \frac{1}{\omega_n}\}$. (Here $\omega_i,\ \omega_{i+1}\ , i = 1, ..., n-1$ denote the minimal, respective maximal sectional osculatory radii at p.) □

Corollary 2. *Let $\Sigma^n, \mathcal{D}$ be as above. If there exists $k_0 > 0$, such that $k(p) \leq k_0$, for all $p \in \Sigma^n$, then there exists a sampling of Σ^n of finite density everywhere. In particular, if Σ^n is compact, then there exists a sampling of Σ^n having uniformly bounded density.*

Proof. Immediate from the theorem above. □

In the special case of surfaces (i.e., $n = 2$), more specific, geometric conditions follow immediately.

Corollary 3. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a smooth surface. In the following cases there exist k_0 as in Corollary 2 above:*

(1) *There exist H_1, H_2, K_1, K_2, such that $H_1 \leq H(p) \leq H_2$ and $K_1 \leq K(p) \leq K_2$, for any $p \in \Sigma^2$, where H, K denote the mean, respective Gauss curvature. (That is both mean and Gauss curvatures are pinched.)*

(2) *The Willmore integrand $W(p) = H^2(p) - K(p)$ and K (or H) are pinched.*

Note that such geometric conditions, are hard to impose in higher-dimension, hence the study of such precise geometric constraints is left for further study.

Remark 5. Obviously, Theorem 3 above is of little relevance for the *space forms* $(\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n)$. Indeed, as noted above, this method is relevant for manifolds considered (by the Nash embedding theorem [19]) as submanifolds of $\mathbb{R}^N$, for some N large enough.

We approach the problem of sampling for nonsmooth manifolds and begin by proposing the following definition.

Definition 16. Let $\Sigma^n, n \geq 2$ be a (connected) manifold of class $\mathcal{C}^0$, and let Σ^n_δ be a smooth δ-approximation to Σ^n. A sampling of Σ^n_δ is called a *δ-sampling of* Σ^n.

Theorem 4. *Let Σ^n be a connected, non-necessarily compact manifold of class $\mathcal{C}^1$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ^n, such that if $\Sigma^n_\delta \to \Sigma^n$ uniformly, and $\mathcal{D}_\delta \to \mathcal{D}$ in the sense of measures, where $\mathcal{D}_\delta$ denote the densities of Σ^n_δ and $\mathcal{D}$ is the density of a smoothing $\widetilde{\Sigma}^n$ of Σ^n.*

Proof. An immediate consequence of Theorem 2 and its proof, and the methods exposed in Section 3.1. □

Corollary 4. *Let Σ^n be a $\mathcal{C}^0$ manifold with finitely many points at which Σ^n fails to be smooth. Then every δ-sampling of a smooth δ-approximation of Σ^n is in fact, a sampling of Σ^n apart of finitely many small neighborhoods of the points where Σ^n is not smooth.*

Proof. From Lemma 2 and Theorem 2 it follows that any such δ-approximation, Σ^n_δ, coincides with Σ^n outside of finitely many such small neighborhoods. □

Remark 6. In order to obtain a better approximation it is advantageous, in this case, to employ Nash‘s method for smoothing, cf. Remark 5 of Section 3 (see [19, 3] for details).

Reconstruction

We use the secant map as defined in Definition 14 in order to reproduce a PL-manifold as a δ-approximation for the sampled manifold. As said in the beginning of Section 2.3 we may now use smoothing in order to obtain a $\mathcal{C}^\infty$ approximation.

5. From Planar to Spherical Representation

As we already noted above, in image and signal processing, communications (e.g., for certain types of antennas), and other branches of information technologies, it is often desirable to map the higher-dimensional signals on $\mathbb{S}^n$. This is convenient since it renders as representation space (or “canvas” for the images under analysis) a compact space, possessing a well known and easy to manipulate isometry group [5].

We propose three alternative methods to obtain the desired representation on $\mathbb{S}^n$. In the first one we first project the signal onto a hyperplane as done in [4], based on Gehring-Väisälä (see Section 2.2 above), and project the flattened signal onto $\mathbb{S}^n$, using the stereographic projection.

The second method follows basically the same route yet, in addition, we suggest to alter the metric on $\mathbb{S}^n$, so that the overall distortion of the composed projection is better controlled. We consider the advantages and weaknesses of the stereographic projection as compared to the use of the intrinsic or chordal metrics on $\mathbb{S}^n$.

The third method is based on Alexander’s method of constructing of branched covering maps onto $\mathbb{S}^n$. We also present a possible application of these method in medical imaging, for the mapping of highly folded tissues, such as the human brain cortex.

5.1 Stereographic Projection

Recall the definition of the stereographic projection from $\widehat{\mathbb{R}}^n \cup \infty$ to $\mathbb{S}^n$ and some of its basic properties that are important for our context.

Let $\bar{p} = (p_1, p_2, \dots, p_{n+1})$ be a point on the n-dimensional unit sphere, $\mathbb{S}^n$, obtained from the point $\bar{x} = (x_1, x_2, \dots, x_n, 0)$ by the stereographic projection then, assuming that the hyperplane plane $(\mathbb{R}^n, 0)$ is tangent to the sphere at the south pole and the center of projection is the north pole of the sphere, we have:

$$(p_1, p_2, \dots, p_{n+1}) = \left(\frac{2x_1}{r^2+1}, \dots, \frac{2x_n}{r^2+1}, \frac{r^2-1}{r^2+1}\right), \tag{5.1}$$

where $r^2 = x_1^2 + \cdots + x_n^2$.

Since the stereographic projection is conformal, composing it with some quasiconformal map w will as well be quasiconformal with the same conformal constant as w. As for the distortion, it follows by elementary considerations, that the local length distortion, as a function of the angle Φ, between a zenith normal vector $(\frac{x_n}{||x_n||})$, and the direction vector to a point p, on $\mathbb{S}^n$, is equal to $\frac{\tan\Phi}{\Phi}$. Therefore, keeping Φ bounded away form $\frac{\pi}{2}$, ensures bounded distortion. This may be achieved in the detriment of globality.

5.2 The Chordal Metric

Definition 17. The *chordal (spherical) metric* q on $\widehat{\mathbb{R}^n}$ is defined as:

$$q(x, y) = \left|\pi(x) - \pi(y)\right|, \tag{5.2}$$

where π is the stereographic projection (see above) and where “$|\cdot|$” denotes, as in Section 2.1, the standard (Euclidean) metric on $\mathbb{R}^n$.

It follows from the definition of π that:

$$\begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, & x \neq \infty \neq y\,; \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}\,. \end{cases} \tag{5.3}$$

Remark 7. $q(x, x^*) = 1$ iff x, x^* are *antipodal points*.

The connection between the chordal metric and the restriction of the Euclidean metric to $\mathbb{S}^n$ is given by:

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \leq q(x,y) \leq \frac{2|x-y|}{(1+|x|)(1+|y|)}\,, \text{ for all } \quad x,y \in \mathbb{R}^n\,.$$

One can proceed to define the chordal metric in an apparently more general manner, by putting:

$$\sigma(\gamma) = \int_\gamma \frac{|dx|}{1+|x|^2}\,, \tag{5.4}$$

where γ is a rectifiable curve in $\mathbb{R}^n$ and by defining the distance between $x, y \in \mathbb{R}^n$ as

$$\sigma(x,y) = \inf_\gamma \sigma(\gamma)\,, \tag{5.5}$$

where the infimum is taken over all the rectifiable paths γ of ends x and y. This definition easily extends to $\widehat{\mathbb{R}^n}$, to obtain a metric on $\widehat{\mathbb{R}^n}$.

However, the metrics q and σ are equivalent. Indeed:

$$\sigma(x,y) = 2\arcsin q(x,y)\,, \text{ for all } \quad x,y \in \widehat{\mathbb{R}^n}\,,$$

and

$$1 \leq \frac{\sigma(x,y)}{q(x,y)} \leq \pi\,, \text{ for all } \quad x \neq y \in \widehat{\mathbb{R}^n}\,.$$

Remark 8. Quasimeromorphic mappings are quasiregular mappings onto $\widehat{\mathbb{R}^n}$, viewed as a Riemannian manifold equipped with the chordal metric.

However, the importance of the chordal (and intrinsic) metric does not reside solely in producing yet another characterization of quasimeromorphic mappings. In fact, it can be shown that the intrinsic metric is the only natural metric on $\mathbb{S}^n$, in the sense that the unit sphere can be characterized purely in terms of this metric (see [6], Chapter VII).

Moreover, it can be shown that, while being conformal, the stereographic projection is not even a local isometry, more precisely one can prove (see, e.g., [5], Prop. 18.4.4.) the following far stronger fact.

Proposition 3. *Let* $U = \text{int}\, U \subset (\mathbb{S}^n, \sigma)$*. Then there exist no isometric map* $f : U \to \mathbb{R}^n$*, i.e., such that* $|f(x) - f(y)| = \sigma(x,y)$*, for every* $x, y \in U$*.*

Remark 9. Of course, for image processing purposes, it is interesting to consider also mappings from $\mathbb{S}^n$ onto itself., viewed, naturally in this context, as restrictions of mappings from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. Here also a number of theoretical results exist. In particular, one can compute the Lipschitz constant of the isometries of $\mathbb{S}^n$, i.e., of certain Möbius transformations, viewed as transformations of $\mathbb{R}^{n+1}$ (see [31]). Moreover, estimates for the distortion in the chordal metric of K-quasiregular mappings of $\mathbb{R}^n$ onto itself have recently been obtained (see [14]).

5.3 The Alexander Method

The simplest technique for constructing quasimeromorphic mappings $f : M^n \to \mathbb{S}^n$, i.e., "geometric" branched coverings of the unit sphere $\mathbb{S}^n$, is an extension to the geometric case of the purely topological construction known as the "*Alexander method*" [1]: One starts by constructing a suitable *chessboard* triangulation of M^n, i.e., such that its simplices satisfy the condition that every $(n-1)$-face is incident to an even number of n-simplices. Since M^n is orientable, a consistent orientation of all the simplices of the triangulation (i.e., such that two given n-simplices having a $(n-1)$-dimensional face in common will have opposite orientations) can be chosen. Then one quasiconformally maps the simplices of the triangulation into $\widehat{\mathbb{R}^n}$ in a chessboard manner: The positively oriented ones onto the interior of the standard simplex in $\mathbb{R}^n$ and the negatively oriented ones onto its exterior. If the dilatations of the quasiconformal maps constructed above are uniformly bounded — a condition that is fulfilled if the simplices of the triangulation are of uniform fatness — then the resulting map will be quasimeromorphic (see [21]).

Note that by a direct application of the Alexander method, it follows from Theorem 1 and its corollaries, that manifolds as those considered there admit quasimeromorphic representations onto $\mathbb{S}^n$.

Remark 10. Fat triangulations are precisely those for which the individual simplices considered in Alexander's method may each be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homothety, with a fixed L.

Remark 11. One can employ the Alexander method in image processing to obtain quasiregular (quasimeromorphic) mappings that accurately represent surfaces exhibiting inherent foldings, e.g., for accurately mapping the human brain cortex.

5.4 A Geometric Compromise: The Injectivity Radius

The simplicity of the Alexander method has as pitfall the extreme folding and branching it generates. It is therefore unsuitable for most imaging purposes, where larger regions of the manifold have to be represented on the sphere, with the least attainable degree of distortion and branching (see, however, Remark 11 below). Thus one would like to ensure, for every point of the manifold, the existence of a neighborhood that admits a nice, one-to-one representation on $\mathbb{S}^n$. Formally, this is done by focusing upon manifolds with *positive injectivity radius,* where the following.

Definition 18. Let $f : M^n \to P^m$ be a mapping, and let $x \in M^n$. The injectivity radius of f at x is defined as:

$$r_f(x) = \sup\{r \geq 0 \,|\, B(x;r) \text{ is embedded}\}\,, \tag{5.6}$$

and the injectivity radius of f is defined to be:

$$r_f = \inf_{x\in M^n} r_f(x)\,. \tag{5.7}$$

For the existence of a positive injectivity radius in the case of quasiconformal mappings of *hyperbolic manifolds* [28] onto $\mathbb{S}^n$, see [22].

Of course, one would like to obtain actual quasiconformal homeomorphisms, i.e., global representations of bounded distortion, onto $\mathbb{S}^n$. In this direction we have the following result.

Theorem 5 ([10, 32]). *Let M^n be a complete Riemannian manifold with finite volume, and let N^n be a simply-connected Riemannian manifold; $n \geq 3$. If $f : M^n \to N^n$ is a locally homeomorphic quasiregular mapping, then f is injective.*

In particular, one has the following corollary.

Corollary 5. *If M^n is compact, then f is a homeomorphism.*

Moreover, for $M^n \equiv \mathbb{R}^n$, one obtains a proof (see [32]) of the following theorem.

Theorem 6 (The Global Homeomorphism Theorem). *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$ *be a locally homeomorphic quasiregular mapping. Then* f *is a global homeomorphism.*

6. Conclusions and Future Study

The methods for mapping a Riemannian manifold into the n-dimensional sphere, introduced in this article, extend previous studies based on the viewpoint that images and other types of data structures should be considered as surfaces and manifolds embedded in higher-dimensional manifolds. In particular, the methods presented in this article are based on the assumption that the signals, surfaces or manifolds are properly sampled in Shannon's sense. This led to consideration of a sampling theorem for Riemannian manifolds. The sampling scheme presented in this article, is based on the ability to triangulate such a manifold by a fat triangulation. This in turn, relies on geometric properties of the manifold and basically on its curvature. The sampling theorems are applicable for signals that may be presented as Riemannian manifolds, a well established viewpoint in signal processing. Considering this viewpoint in rigorous manner still remains as a challenge for further study. It is common, for instance, to consider a color image as a surface in $\mathbb{R}^5$ yet, it is more prone and probably more accurate to consider it as a three-dimensional manifold embedded in some higher-dimensional Euclidian space. Another direction left for future study is the extent in which the sampling scheme presented herein can be degenerate to one-dimensional signals as an alternative to the classical sampling theorem of Shannon. Some relevant results are already at hand. For example, the methods based on previous studies of the authors on mapping of signals and surfaces into hypersurfaces in $\mathbb{R}^n$ [4], and on sampling of manifolds ([24], [25]), come along with an algorithm readily available for implementation. Other theoretical and applied facets of this problem are currently under investigation.

Acknowledgments

Emil Saucan is supported by the Viterbi Postdoctoral Fellowship. Research is partly supported by the Ollendorf Minerva Center and by the Fund for Promotion of Research at the Technion.

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Received September 29, 2006

Revision received May 23, 2007

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