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# The Finite Zak Transform: An Efficient Tool for Image Representation and Analysis[1]

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**A mathematical approach to image representation and analysis is presented. The formalism is based on the finite Zak transform (FZT), which provides an important tool for the analysis of images that by their very nature are spatially nonstationary. The discrete Zak transform is extended to two spatial dimensions, and fundamental properties of the two-dimensional FZT are discussed, emphasizing the direct relationship that exists between the Zak transform and the Cooley–Tukey FFT algorithm. Subsampling and interpolation are considered in the context of various mappings into the combined space. The two-dimensional FZT is applied to image representation and image analysis in computation of the Gabor expansion coefficients with arbitrary resolution. A technique for stable reconstruction is implemented and illustrated.** © 1992 Academic Press, Inc.

## 1. INTRODUCTION

Out of the many studies conducted over the last few decades in the areas of image structure understanding and vision there emerged several basic interrelated findings which can be summarized as follows: (1) A wide spectrum of natural images obeys relatively simple fractal models which depict self-similarity under magnification [1], and are suggestive as such of a multiresolution approach to image representation and analysis [2, 3]. (2) Both natural scenes and man-made environments, which often are composed of a mosaic of stochastically and/or deterministically textured segments of diverse nature, are nonstationary [7, 8]. (3) The majority of images acquired from our visual environment are neither globally periodic (as is the case with a uniformly textured visual field) nor fully localized. Since natural images are composed of both periodic and discretely localized information, they are most efficiently represented by a scheme which incorporates aspects of both spatial (i.e., positional information) and spatial frequency approaches to image representation [8, 7, 9, 4, 11, 5, 6].

Several advantages are inherent in the scheme of image representation in the combined position–frequency space, and it is often desirable to apply a technique wherein local features in the form of frequency signatures are confined to an effective small portion of the entire visual field. For example, one goal of image representation and analysis is to project and process the image so that desired objects or elements of the scene stand out well above the masking background made up of clutter and noise [5]. The purpose of image processing and further analysis is, in this case, to estimate information-bearing parameters, and to separate attributes which are characteristic of a desired object from disturbances, cluttered background, and "false images" introduced by subsampling and quantization effects. Clearly, analysis of such images and visual scenarios cannot be accomplished by application of spatial-domain representations such as correlation methods or by frequency-domain representations based on Fourier transform. Such classical techniques are based on the assumption that information distribution over the visual field and the contaminating noise are stationary. Further, although spatial-domain techniques provide the required resolution, the local distribution of energy per se may often not suffice for clear detection and identification of a weak signal, and neither is the global Fourier transform adequate for highlighting a desired localized object. In such circumstances it appears to be desirable to combine the spatial and spatial frequency approaches for the representation and analysis of visual images [8, 4, 11, 5, 6]. Although a representation in

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the combined position–frequency space entails an increase in the dimensionality, the efficient projection of an image onto a small cluster of expansion coefficients presents also an interesting approach to data reduction and compression in cases in which a lossy image reconstruction with good (subjective) quality is permitted [11, 8, 5, 6]. Several techniques are suitable for signal representation in the combined space. Of these, the Wigner–Ville distribution [21] and the ambiguity function [20], two of the more widely used representations in the combined space, are based on bilinear transformations. As such they present a problem when several objects are distributed, and have to be separated, over the visual field. Wavelet-type transforms [22, 23] and the Gaborian pyramid as such [11], have more recently been applied in image analysis and representation. The Zak transform (ZT), adopted in the context of the present study, is a linear transformation [20]. It maps a two-dimensional data set (i.e., an image) onto a fundamental four-dimensional cube and provides, as such, position–frequency information. The ZT is central to the theory and application of the Gabor representation [12, 13]. It highlights interesting and powerful characteristics inherent in the Gabor expansion. It also permits the accomplishment of finite representation in the combined space, and lends itself to efficient means of computational implementation [5, 6, 13].

The finite Zak transform (FZT) is first defined, and its fundamental properties are presented. The direct relation of the FZT to the Cooley–Tukey FFT algorithm is then established. The FZT is implemented in computation of the Gabor expansion coefficients, illustrating the stability and computational efficiency of the algorithm which is directly derived from the fundamental relationship that exists between the ZT and the FFT.

## 2. THE FINITE ZAK TRANSFORM OF IMAGES

Digital image processing is in practice applicable to finite two-dimensional data sets. For the sake of simplicity, we assume that the data are defined over the integer pairs $(n_x, n_y) \in Z \times Z$ and subject to the following periodicity conditions. Given a positive integer $N$, a function $f(n_x, n_y), n_x, n_y \in Z$, is called $N \times N$-periodic if it satisfies

$$f(n_x + N, n_y + N) = f(n_x, n_y), \quad (n_x, n_y) \in Z \times Z. \quad (1)$$

An $N \times N$-periodic function $f$ is determined by its values

$$f(n_x, n_y), \quad (n_x, n_y) \in Z \times Z, \quad (2)$$

and will be called $N \times N$-image data. Denote $\mathscr{L}^2(Z/N \times Z/N)$ the Hilbert space of all $N \times N$-periodic and square summable functions with inner product of such functions being defined by

$$\langle f, g \rangle = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{N-1} f(n_x, n_y) g^*(n_x, n_y), \quad f, g \in \mathscr{L}^2(Z/N \times Z/N). \quad (3)$$

For image data sets satisfying the periodicity condition (1), we define the FZT as follows:

DEFINITION 1. For any divisor $L$ of $N$, the FZT $(\mathscr{Z}_L f)(i, j; \rho, \sigma), i, j; \rho, \sigma \in Z$, of an image $f \in \mathscr{L}^2(Z/N \times Z/N)$ *is defined by*

$$(\mathscr{Z}_L f)(i, j; \rho, \sigma) = \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} f(i + Mr, j + Mp) \exp\left(2\pi\iota \frac{\rho r}{L}\right) \exp\left(2\pi\iota \frac{\sigma p}{L}\right), \quad N = L \cdot M. \quad (4)$$

It follows directly from (4) that $\mathscr{Z}_L f$ is $N \times N$-periodic in each pair of variables.

To highlight additional features of the ZT, we present the following theorem:

THEOREM 1. *For* $f \in \mathscr{L}^2(Z/N \times Z/N), N = L \cdot M$, *the ZT has the following readily discernible periodicity qualtities:*

$$(\mathscr{Z}_L f)(i + M, j + M; \rho, \sigma) = \exp\left(-2\pi\iota \frac{\rho}{L}\right) \exp\left(-2\pi\iota \frac{\sigma}{L}\right) (\mathscr{Z}_L f)(i, j; \rho, \sigma), \quad (5)$$

$$(\mathscr{Z}_L f)(i, j; \pi + L, \sigma + L) = (\mathscr{Z}_L f)(i, j; \rho, \sigma), \quad i, j \in Z. \quad (6)$$

*Proof.*

$$\begin{aligned}(\mathscr{Z}_L f)(i + M, j + M; \rho, \sigma) &= \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} f(i + M + Mr, j + M + Mp) \exp\left(2\pi\iota \frac{\rho r}{L}\right) \exp\left(2\pi\iota \frac{\sigma p}{L}\right) \\ &= \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} f(i + Mr, j + Mp) \exp\left(2\pi\iota \frac{\rho(r-1)}{L}\right) \exp\left(2\pi\iota \frac{\sigma(p-1)}{L}\right) \\ &= \exp\left(-2\pi\iota \frac{\rho + \sigma}{L}\right) (\mathscr{Z}_L f)(i, j; \rho, \sigma).\end{aligned}$$

The second quality follows directly from the $L$-periodicity of $\exp(2\pi\iota(r\sigma/L))$ as a function of $\sigma$. ■

Thus, $\mathfrak{Z}_L f(i, j; \rho, \sigma)$ is completely determined by its values on a four-dimensional cube of size $(M \times M) \times (L \times L)$.

The ZT is actually a two-dimensional Fourier transform (FT) of size $L \times L$ of the values

$$f(i_0, j_0), f(i_0 + M, j_0 + M), \ldots, f(i_0 + (L-1)M, j_0 + (L-1)M), \quad (7)$$

for each point $(i_0, j_0) \in M \times M$. Thus, the computation of the ZT $\mathfrak{Z}_L f$ in this way entails $M^2$ two-dimensional FTs of a smaller size $L \times L$.

The data set $f(i, j)$, $0 \le i, j < M$, can be recovered from its ZT by the formula

$$f(i,j) = L^{-2} \sum_{\rho=0}^{L-1} \sum_{\sigma=0}^{L-1} (\mathfrak{Z}_L f)(i, j; \rho, \sigma), \quad 0 \le i, j < M. \quad (8)$$

Thus, by using (5) and (8) an image data set can be fully recovered.

For each $0 \le i, j < N$ we can view the representation (4) as a Fourier series in $\mathcal{L}^2(Z/L \times Z/L)$. The Fourier coefficients are given by the image data set (7). The inversion formula depends on the values of the FZT of the image. By Fourier inversion of (4) we have

$$f(i + Mr, j + Mp) = L^{-2} \sum_{\rho=0}^{L-1} \sum_{\sigma=0}^{L-1} (\mathfrak{Z}_L f)(i, j; \rho, \sigma) \exp\left(-2\pi\iota \frac{\rho r + \sigma p}{L}\right), \quad 0 \le i, j < M, 0 \le r, p < L. \quad (9)$$

Any part of the image can be reconstructed by weighted summation of the ZT. The qualities (5) and (6) of the ZT which constitute Theorem 1 completely characterize the space of FZT corresponding to $L$.

The following theorem states conditions required of a function $F(i, j; \rho, \sigma)$ to be a ZT.

THEOREM 2. *Suppose $F(i, j; \rho, \sigma)$, $i, j, \rho, \sigma \in Z$, is a complex valued function satisfying*

$$F(i + M, j + M; \rho, \sigma) = \exp\left(-2\pi\iota \frac{\rho + \sigma}{L}\right) F(i, j; \rho, \sigma), \quad (10)$$

$$F(i, j; \rho + L, \sigma + L) = F(i, j; \rho, \sigma), \quad i, j, \rho, \sigma \in Z, \quad (11)$$

*for some positive integers $L$ and $M$. Then, the function*

$$f(i,j) = L^{-2} \sum_{\rho=0}^{L-1} \sum_{\sigma=0}^{L-1} F(i, j; \rho, \sigma), \quad i, j \in Z, \quad (12)$$

*is in $\mathcal{L}^2(Z/N \times Z/N)$, $N = L \cdot M$, and*

$$F(i, j; \rho, \sigma) = (\mathfrak{Z}_L f)(i, j; \rho, \sigma), \quad i, j, \rho, \sigma \in Z. \quad (13)$$

*Proof.* Condition (10) implies that $F$ is $N$-periodic in the variables $i$ and $j$, i.e.,

$$\begin{aligned} F(i + N, j + N; \rho, \sigma) &= F(i + (L-1)M + M, j + (L-1)M + M; \rho, \sigma) \\ &= \exp\left(-2\pi\iota \frac{\rho+\sigma}{L}\right) F(i + (L-1)M, j + (L-1)M; \rho, \sigma) \\ &= \exp\left(-2\pi\iota \frac{2(\rho+\sigma)}{L}\right) F(i + (L-2)M, j + (L-2)M; \rho, \sigma) \\ &\;\;\vdots \\ &= \exp\left(-2\pi\iota \frac{L(\rho+\sigma)}{L}\right) F(i, j; \rho, \sigma) \\ &= F(i, j; \rho, \sigma) \end{aligned}$$

Thus, $f(i, j)$ is $N \times N$-periodic.

Since $F$ and $\mathfrak{Z}_L f$ satisfy conditions (10) and (11), we need to prove (13) for $0 \le i, j < M$, $0 \le \rho, \sigma < L$. Inserting (12) into the definition of the FZT,

$$\begin{aligned} (\mathfrak{Z}_L f)(i, j; \rho, \sigma) &= \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} f(i + Mr, j + Mp) \exp\left(2\pi\iota \frac{r\rho + p\sigma}{L}\right) \\ &= L^{-2} \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} \sum_{\gamma=0}^{L-1} \sum_{\delta=0}^{L-1} F(i + rM, j + pM; \gamma, \delta) \exp\left(2\pi\iota \frac{r\rho + p\sigma}{L}\right), \end{aligned}$$

we have, for $0 \le i, j < M$, $0 \le \rho, \sigma < L$, by (10)

$$(\mathfrak{Z}_L f)(i, j; \rho, \sigma) = L^{-2} \sum_{\gamma=0}^{L-1} \sum_{\delta=0}^{L-1} F(i, j; \gamma, \delta) \sum_{r=0}^{L-1} \sum_{p=0}^{L-1} \exp\left(2\pi\iota \frac{p(\sigma - \delta)}{L}\right) \exp\left(2\pi\iota \frac{r(\rho - \gamma)}{L}\right) = F(i, j; \rho, \sigma),$$

since

$$L^{-1}\sum_{r=0}^{L-1}\exp\left(2\pi\iota\frac{r(\rho-\gamma)}{L}\right)=\begin{cases}1, & \rho=\gamma,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$L^{-1}\sum_{p=0}^{L-1}\exp\left(2\pi\iota\frac{p(\sigma-\delta)}{L}\right)=\begin{cases}1, & \sigma=\delta,\\ 0, & \text{otherwise.}\end{cases} \blacksquare$$

Denote by $\mathcal{L}^2(\mathfrak{X}_L)$ the Hilbert space of all complex valued functions $F(i, j; \rho, \sigma)$, $i, j, \rho, \sigma \in Z$, satisfying conditions (10) and (11) with inner product defined by

$$\langle F, G\rangle=\sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{\rho=0}^{L-1}\sum_{\sigma=0}^{L-1}F(i, j; \rho, \sigma)G^*(i, j; \rho, \sigma). \tag{14}$$

The following theorem follows directly from the definitions.

THEOREM 3. *The mapping $L^{-1/2}\mathfrak{X}_L$ is an isometry from $\mathcal{L}^2(Z/N \times Z/N)$ onto $\mathcal{L}^2(\mathfrak{X}_L)$. If $f \in \mathcal{L}^2(Z/N \times Z/N)$ and $F = \mathfrak{X}_L f$, then*

$$\|f\|^2 = L^{-2}\|\mathfrak{X}_L f\|^2 = L^{-2}\|F\|^2. \quad \blacksquare \tag{15}$$

COROLLARY 1. *Let $f$, $g$ be two image data sets and $\mathfrak{X}_L f$, $\mathfrak{X}_L g$ their ZTs, respectively; then*

$$\langle \mathfrak{X}_L f, \mathfrak{X}_L g\rangle = \langle f, g\rangle. \tag{16}$$

## 3. THE FZT AND COOLEY–TUKEY FFT

The FZT is the main building block of the Cooley–Tukey FFT algorithm [19]. To substantiate this observation we first prove the following fundamental result.

THEOREM 4. *If $f \in \mathcal{L}^2(Z/N \times Z/N)$ and $F$ is its $N \times N$-point FT, then*

$$\exp\left(-2\pi\iota\frac{\rho i+\sigma j}{N}\right)(\mathfrak{X}_L f)(i, j; -\rho, -\sigma) = M^{-2}(\mathfrak{X}_M F)(\rho, \sigma; i, j). \tag{17}$$

*Proof.* Consider the image as $f(i + Mr, j + Mp)$, $i, j = 0, 1, \ldots, M - 1$, $r, p = 0, 1, \ldots, L - 1$. Its two-dimensional FT, $F(\rho + L\gamma, \sigma + L\delta)$, $\rho, \sigma = 0, 1, \ldots L - 1$, $\gamma, \delta = 0, 1, \ldots, M - 1$, can be written as

$$F(\rho + L\gamma, \sigma + L\delta)$$

$$= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\exp\left(-2\pi\iota\frac{\rho i+\sigma j}{N}\right)\exp\left(-2\pi\iota\frac{\gamma i+\delta j}{M}\right)$$

$$\sum_{r=0}^{L-1}\sum_{p=0}^{L-1}f(i+Mr, j+Mp)\exp\left(-2\pi\iota\frac{\rho i+\sigma p}{L}\right) \tag{18}$$

$$= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}(\mathfrak{X}_L f)(i, j; -\rho, -\sigma)\exp\left(-2\pi\iota\frac{\rho i+\sigma j}{N}\right)$$

$$\exp\left(-2\pi\iota\frac{\gamma i+\delta j}{M}\right).$$

The proof is completed by taking the ZT of both sides of (18). ■

Theorem 4 shows that the ZT of an image contains spatial–frequency information. Loosely speaking, formula (17) asserts that up to the phase factor $\exp(-2\pi\iota(\rho i + \sigma j)/N))$, the ZT $\mathfrak{X}_L f$ produces the ZT $\mathfrak{X}_M F$. The absolute value of the image and its FT are identical, neglecting permutation. The Cooley–Tukey FFT algorithm can be formulated in terms of the ZT, formula (17). The FT $F$ can be computed by the following sequence of steps:

1. Compute $(\mathfrak{X}_L f)(i, j; \rho, \sigma)$, $0 \le i, j < M$, $0 \le \rho, \sigma < L$, and take a mirror image in the frequency domain.
2. Multiply by a phase factor $\exp(-2\pi\iota(\rho i + \sigma j)/N))$. This step requires $N \times N$ multiplications as $0 \le i, j < M$, $0 \le \rho, \sigma < L$. The result is the ZT of the FT of the original image.
3. Compute the inverse ZT $\mathfrak{X}_M^{-1}$ by (9) with $L$ and $M$ interchanged. This step requires $L^2$ two-dimensional DFTs of size $M \times M$.

In other words, Theorem 4 states that the ZT is a partial result of the FFT algorithm.

## 4. GABORIAN WAVELETS

### 4.1. *Basis $\mathcal{L}^2(Z/N \times Z/N)$*

Let $g \in \mathcal{L}^2(Z/N \times Z/N)$ and define for $m, n, k, l \in Z/N$ the function $g_{mnkl} \in \mathcal{L}^2(Z/N \times Z/N)$ by the formula

$$g_{mnkl}(i, j) = g(i + m, j + n)\exp\left(-2\pi\iota\frac{ik+jl}{N}\right), \quad i, j \in Z/N. \tag{19}$$

We call the functions $\{g_{mnkl}(i, j)\}$, $m, n, k, l \in Z/N$, the *Gaborian wavelets* [15], or simply wavelets, and $g$ the generator of the wavelets. For fixed $g$, we will use the FZT to study properties of the wavelets corresponding to $g$.

THEOREM 5. *For $N = L \cdot M$, $m, n, k, l \in Z/N$, and a given generator $g$,*

$$(\mathfrak{Z}_L g_{mnkl})(i, j; \rho, \sigma) = \exp\left(-2\pi\iota \frac{ik + jl}{N}\right)(\mathfrak{Z}_L g)(i + m, j + n; \rho - k, \sigma - l). \quad (20)$$

*Proof.* The left-hand side of (20) is

$$\begin{aligned}(\mathfrak{Z}_L g_{mnkl})(i, j; \rho, \sigma) &= \sum_{r=0}^{L-1}\sum_{p=0}^{L-1} g_{mnkl}(i + rM, j + pM)\exp\left(2\pi\iota \frac{r\rho + p\sigma}{L}\right)\\ &= \sum_{r=0}^{L-1}\sum_{p=0}^{L-1} g(i + m + rM, j + n + pM)\\ &\quad \exp\left(2\pi\iota \frac{r\rho + p\sigma}{L}\right)\\ &\quad \exp\left(-2\pi\iota \frac{(i + rM)k + (j + pM)l}{N}\right)\\ &= \exp\left(-2\pi\iota \frac{ik + jl}{N}\right)\\ &\quad \sum_{r=0}^{L-1}\sum_{p=0}^{L-1} g(i + m + rM, j + n + pM)\\ &\quad \exp\left(2\pi\iota \frac{r(\rho - k)M}{N}\right)\exp\left(2\pi\iota \frac{p(\sigma - l)M}{N}\right)\\ &= \exp\left(-2\pi\iota \frac{ik + jl}{N}\right)(\mathfrak{Z}_L g)\\ &\quad \times (i + m, j + n; \rho - k, \sigma - l),\end{aligned}$$

proving the theorem. ■

A special interesting case is obtained when $m = m'M$, $n = n'M$, and $k = k'L$, $l = l'L$.

COROLLARY 2. *For* $0 \le m', n' < L$, $0 \le k', l' < M$,

$$(\mathfrak{Z}_L g_{m'M,n'M,k'L,l'L})(i, j; \rho, \sigma) = (\mathfrak{Z}_L g)(i, j; \rho, \sigma) \times \exp\left(-2\pi\iota\left(\frac{m'\rho}{L} + \frac{n'\sigma}{L}\right)\right)\exp\left(-2\pi\iota\left(\frac{k'i}{M} + \frac{l'j}{M}\right)\right). \quad (21)$$

*Proof.* According to Theorem 5, the left-hand side of (21) is

$$\exp\left(-2\pi\iota \frac{ik'L + jl'L}{N}\right)(\mathfrak{Z}_L g)(i + m'M, j + n'M; \rho - k'L, \sigma - l'L),$$

which by Theorem 1 is

$$\begin{aligned}&\exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right)(\mathfrak{Z}_L g)(i + m'M, j + n'M; \rho - k'L, \sigma - l'L)\\ &= \exp\left(-2\pi\iota\left(\frac{m'\rho}{L} + \frac{n'\sigma}{L}\right)\right)\exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right)(\mathfrak{Z}_L g)(i, j; \rho, \sigma),\end{aligned}$$

proving the corollary. ■

We now proceed to determine the necessary and sufficient conditions for the Gaborian wavelet (19) to be the basis of $L^2(Z/N \times Z/N)$.

THEOREM 6. *The set*

$$\{g_{m'M,n'M,k'L,l'L} : 0 \le m', n' < L, 0 \le k', l' < M\} \quad (22)$$

*constitutes a basis of* $L^2(Z/N \times Z/N)$ *if and only if* $\mathfrak{Z}_L g$ *does not vanish.*

*Proof.* Suppose for all $0 \le m', n' < L$, $0 \le k', l' < M$,

$$\langle f, g_{m'M,n'M,k'L,l'L}\rangle = 0.$$

By Theorem 3 and the previous corollary

$$\langle \mathfrak{Z}_L f, \mathfrak{Z}_L g_{m'M,n'M,k'L,l'L}\rangle = 0,$$

which can be rewritten as

$$\sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{\rho=0}^{L-1}\sum_{\sigma=0}^{L-1}(\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j; \rho, \sigma) \times \exp\left(-2\pi\iota\left(\frac{m'\rho}{L} + \frac{n'\sigma}{L}\right)\right)\exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right) = 0.$$

The summation is a four-dimensional $(M \times L) \times (M \times L)$ FT. It therefore follows that the products

$$(\mathfrak{Z}_L f)(i, j, \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j, \rho, \sigma) = 0,$$
$$0 \le i, j < M, 0 \le \rho, \sigma < L.$$

According to Theorem 1 this holds for all $i, j, \rho, \sigma \in Z$.

If $\mathfrak{Z}_L g$ never vanishes, $Z_L f$ is identically zero. Consequently, $f$ is identically zero and the wavelets (22) span $L^2(Z/N \times Z/N)$. By dimension they form a basis of $L^2(Z/N \times Z/N)$.

Conversely, if $\mathfrak{Z}_L g$ vanishes somewhere, then a nontrivial $f$ can be found such that all products $(\mathfrak{Z}_L f)(i, j, \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j, \rho, \sigma) = 0$. Reversing the steps of the preceding argument, $f$ is a nontrivial function not contained in the linear span of the wavelets of (22), implying, in turn, that this wavelet set is not a spanning set, completing the proof of the theorem. ■

### 4.2. *Inner Products*

An inner product of an image and a template is an important operation in image processing, pattern recognition, and target detection. It is of special interest to derive a fast algorithm for computation of the inner product of an image and the wavelet basis [7, 9, 10]. Consider the product function

$$(\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j; \rho, \sigma) \quad i, j, \rho, \sigma \in Z. \tag{23}$$

It is easy to see that (23) is $M$-periodic according to Theorem 1 in the variables $i, j$ and $L$-periodic in the variables $\rho, \sigma$.

From Corollaries 1 and 2

$$\begin{aligned}\langle f, g_I\rangle &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{\rho=0}^{L-1}\sum_{\sigma=0}^{L-1} (\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g_I)(i, j; \rho, \sigma)\\ &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{\rho=0}^{L-1}\sum_{\sigma=0}^{L-1} (\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j; \rho, \sigma)\\ &\quad \times \exp\left(+2\pi\iota\left(\frac{m'\rho}{L} + \frac{n'\sigma}{L}\right)\right) \exp\left(+2\pi\iota\,\frac{ik' + jl'}{M}\right),\end{aligned} \tag{24}$$

where $I = \{m'M, n'M, k'L, l'L\}$. The result presented by (24) expresses the product (23) as the inverse four-dimensional $(M \times L) \times (M \times L)$ FT of the array

$$[\langle f, g_I\rangle]_{0\le k',l'<M, 0\le m',n'<L}. \tag{25}$$

The product (23) can be interpreted as Fourier coefficients of the inner product of an image and a template. Thus, the algorithm for computing the inner product of the image and a Gaborian wavelet can be stated as the following sequence of steps:

1. Compute the ZT $\mathfrak{Z}_L g$ of the wavelet $g$ and store it.
2. Compute the ZT of the image data set $\mathfrak{Z}_L f$.
3. Take an inverse discrete FT of the product

$$\begin{aligned}\langle f, g_I\rangle &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\sum_{\rho=0}^{L-1}\sum_{\sigma=0}^{L-1} (\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g)(i, j; \rho, \sigma)\\ &\quad \times \exp\left(+2\pi\iota\left(\frac{m'\rho}{L} + \frac{n'\sigma}{L}\right)\right) \exp\left(+2\pi\iota\,\frac{ik' + jl'}{M}\right).\end{aligned} \tag{26}$$

The array (25), where each of its elements is obtained by (26), defines the Fourier coefficients of the expansion of the product function (23).

### 4.3. *Wavelet Bases for Refined Grids*

The idea is to refine the grid, and corresponding resolution of the expansion, by overlaying a set of shifted grids. The scheme is accordingly defined as follows:

Take

$$\begin{aligned} m &= m'' + m'M, & 0 \le m'' < M, 0 \le m' < L,\\ n &= n'' + n'M, & 0 \le n'' < M, 0 \le n' < L,\\ k &= k'' + k'M, & 0 \le k'' < L, 0 \le k' < M,\\ l &= l'' + l'M, & 0 \le l'' < L, 0 \le l' < M.\end{aligned} \tag{27}$$

Direct computation shows that

$$\begin{aligned} g_{mnkl}(i, j) &= g(i + m'' + m'M, j + n'' + n'M)\\ &\quad \times \exp\left(-2\pi\iota\,\frac{i(k'' + k'L) + j(l''L)}{N}\right)\\ &= g_{m''n''k''l''}(i + m'M, j + n'M)\\ &\quad \times \exp\left(-2\pi\iota\,\frac{(ik' + jl')L'}{N}\right) \exp\left(+2\pi\iota\,\frac{m'k'' + n'l''}{L}\right)\\ &= (g_{m''n''k''l''})_I(i, j) \exp\left(+2\pi\iota\,\frac{m'k'' + n'l''}{L}\right).\end{aligned} \tag{28}$$

Replacing $g$ by $g_{m'',n''}$ in the preceding discussion, and applying Corollary 2, we have the following.

THEOREM 7. *For* $m, n, \rho, \sigma \in Z/N$, *defined according to the scheme of* (27),

$$\begin{aligned}(\mathfrak{Z}_L g_{mnkl})(i, j; \rho, \sigma) &= \exp\left(2\pi\iota\,\frac{m'k'' + n'l''}{L}\right) (\mathfrak{Z}_L g_{m''n''k''l''})_{m'M,n'M,k'L,l'L}\\ &\quad \times (i, j; \rho, \sigma)\\ &= \exp\left(2\pi\iota\,\frac{m'k'' + n'l''}{L}\right) (\mathfrak{Z}_L g_{m''n''k''l''})(i, j; \rho, \sigma)\\ &\quad \times \exp\left(-2\pi\iota\,\frac{(m'\rho + n'\sigma)}{L}\right) \exp\left(-2\pi\iota\,\frac{ik' + jl'}{M}\right).\end{aligned} \tag{29}$$

The product function

$$(\mathfrak{Z}_L f)(i, j; \rho, \sigma)(\mathfrak{Z}_L^* g_{m''n''k''l''})(i, j; \rho, \sigma) \tag{30}$$

is $M \times M$ periodic in the variables $i, j$ and $L \times L$-periodic in the variables $\rho, \sigma$. Its Fourier series expansion is

$$\sum_{m'=0}^{L-1}\sum_{n'=0}^{L-1}\sum_{k'=0}^{M-1}\sum_{l'=0}^{M-1}\langle f, (g_{m''n''k''l''})_{m'M,n'M,k'L,l'L}\rangle$$
$$\times \exp\left(+2\pi\iota\left(\frac{(m'\rho}{L}+\frac{n'\sigma)}{L}\right)\right)\exp\left(+2\pi\iota\frac{ik'+jl'}{M}\right). \tag{31}$$

It follows that for fixed $0 \le m'', n'' < M$, $0 \le k'', l'' < L$, the set of wavelets

$$\{g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} : 0 \le m', n' < L, \quad 0 \le k', l' < M\} \tag{32}$$

constitutes a basis of $L^2(Z/N \times Z/N)$ *if and only if* $(\mathfrak{Z}_L g_{m'',n'',k'',l''})$ never vanishes. In light of Theorem 5, this is equivalent to the requirement that $\mathfrak{Z}_L g$ never vanishes.

It is always the case that for nontrivial $g$, the set of wavelets

$$\{g_{mnkl} : 0 \le m, n, k, l < N\} \tag{33}$$

spans $L^2(Z/N \times Z/N)$.

## 5. FINITE GABOR EXPANSION OF IMAGES

In the present study we assume that the image $f(i, j)$ contains $N \times N$ pixels and is square integrable. Let $g_{mnkl}(i, j)$ be a family of Gaborian wavelets of the type defined earlier (19). Represent the image and the wavelets as four-dimensional arrays of smaller dimension. For $N = M \cdot L$,

$$r, s = 0, 1, \ldots, L-1 \quad \text{and} \quad i, j = 0, 1, \ldots, M-1,$$

we denote

$$f(i, r; j, s) = f(i + rM, j + sM),$$

and

$$g_{mnkl}(i, r; j, s) = g_{mnkl}(i + rM, j + sL).$$

The FZT is then represented as

$$\mathfrak{Z}_L f(i, j; \rho, \sigma) = \sum_{r=0}^{L-1}\sum_{s=0}^{L-1} f(i + rM, j + sM) \times \exp\left(-2\pi\iota\frac{r\rho + s\sigma}{L}\right),$$
$$0 \le i, j \le M-1, \quad 0 \le \rho, \sigma \le L-1.$$

A finite Gabor expansion of an image is a linear combination of wavelets:

$$f(i, r; j, s) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} c_{m,n,k,l}\, g_{mnkl}(i, r; j, s). \tag{34}$$

The coefficients $c_{m,n,k,l}$, $0 \le m, n, k, l < N$, are called Gabor coefficients of the expansion. Since the set of Gaborian wavelets is not linearly independent, $f$ admits several Gabor expansions. In contrast, the inner product $\langle f, g_{mnkl}\rangle$, $0 \le m, n, k, l < N$, provides a unique measure of the contribution of $g_{mnkl}$ to the representation of $f$.

To compute the Gabor coefficients select a factorization $N = L \cdot M$ which determines the complementary resolutions along the frequency and spatial axes, respectively. Take

$$m = m'M, \quad n = n'M, \quad k = k'L, \quad l = l'L.$$

Then, provided $\mathfrak{Z}_L g$ never vanishes, according to Theorem 6, every $f \in L^2(Z/N \times Z/N)$ can be represented *uniquely* in the form

$$f(i, r; j, s) = \sum_{m'=0}^{L-1}\sum_{n'=0}^{L-1}\sum_{k'=0}^{M-1}\sum_{l'=0}^{M-1} c_{m'M,n'M,k'L,l'L}\, g_{m'M,n'M,k'L,l'L}(i, r; j, s). \tag{35}$$

The coefficients $c_{m'M,n'M,k'L,l'L}$, $0 \le m', n' < L$, $0 \le k', l' < M$, are called the Gabor coefficients of the image $f$ corresponding to the wavelet $g_{m'M,n'M,k'L,l'L}$.

Applying the FZT to both sides of (35), we have according to Theorem 5

$$(\mathfrak{Z}_L f)(i, j; \rho, \sigma) = (\mathfrak{Z}_L g)(i, j; \rho, \sigma) \times \sum_{m'=0}^{L-1}\sum_{n'=0}^{L-1}\sum_{k'=0}^{M-1}\sum_{l'=0}^{M-1} c_{m'M,n'M,k'L,l'L}$$
$$\times \exp\left(-2\pi\iota\left(\frac{(m'\rho}{L}+\frac{n'\sigma)}{L}\right)\right)\exp\left(-2\pi\iota\left(\frac{k'i}{M}+\frac{l'j}{M}\right)\right). \tag{36}$$

The Gabor coefficients

$$c_{m'M,n'M,k'L,l'L}, \quad 0 \le m', n' < L, 0 \le k', l' < M, \tag{37}$$

of the image $f$, corresponding to the basis $g_{m'M,n'M,k'L,l'L}(i, r; j, s)$, $0 \le m', n' < L$, $0 \le k', l' < M$, can be computed by the inverse $(M \times L) \times (M \times L)$ four-dimensional FT of the quotient function

$$\frac{(\mathfrak{Z}_L f)(i, j; \rho, \sigma)}{(\mathfrak{Z}_L g)(i, j; \rho, \sigma)}, \quad 0 \le i, j < M, 0 \le \rho, \sigma < L. \tag{38}$$

We proceed to show that the Gabor coefficients are related to the inner product of the image and the Gaborian wavelet. From (36) we have

$$(\mathscr{Z}_L f)(i, j; \rho, \sigma)(\mathscr{Z}_L^* g)(i, j; \rho, \sigma) = |\mathscr{Z}_L g(i, j; \rho, \sigma)|^2 \times \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} c_{m'M,n'M,k'L,l'L} \times \exp\left(-2\pi\iota \left(\frac{(m'\rho}{L} + \frac{n'\sigma)}{L}\right)\right) \exp\left(-2\pi\iota \left(\frac{k'i}{M} + \frac{l'j}{M}\right)\right). \quad (39)$$

The left-hand side of (39), according to (24), is also equal to

$$(\mathscr{Z}_L f)(i, j; \rho, \sigma)(\mathscr{Z}_L^* g)(i, j; \rho, \sigma) = \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} \langle f, g_I \rangle \exp\left(+2\pi\iota \left(\frac{(m'\rho}{L} + \frac{n'\sigma)}{L}\right)\right) \times \exp\left(+2\pi\iota \left(\frac{(ik' + jl')}{M}\right)\right). \quad (40)$$

Since

$$|(\mathscr{Z}_L g)(i, j; \rho, \sigma)|^2 = \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} \langle g, g_I \rangle \times \exp\left(+2\pi\iota \left(\frac{(m'\rho}{L} + \frac{n'\sigma)}{L}\right)\right) \exp\left(+2\pi\iota \left(\frac{(ik' + jl')}{M}\right)\right), \quad (41)$$

combining (39), (40), and (41) we can write

$$\langle f, g_I \rangle = c_I ** \langle g, g_I \rangle, \quad (42)$$

where ** denotes the four-dimensional convolution.

## 6. COMPUTATION OF GABOR COEFFICIENTS WITH ARBITRARY SPECIFIED RESOLUTION

Consider the "coarse" basis of $L^2(Z/N \times Z/N)$

$$\{g_{m'M,n'M,k'L,l'L}, \quad 0 \le m', n' < L, 0 \le k', l' < M\}. \quad (43)$$

For any fixed $m''$, $n''$ in the range $0 \le m'' < M$, $0 \le n'' < L$, the "shifted" bases of $L^2(Z/N \times Z/N)$ are

$$\{g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} : 0 \le m', n' < L, \quad 0 \le k', l' < M\}. \quad (44)$$

We next define Gabor coefficients with respect to the shifted basis. Fix any $m''$, $n''$, $k''$, $l''$ in the range $0 \le m''$, $n'' < M$, $0 \le k''$, $l'' < L$. Then any image $f \in L^2(Z/N \times Z/N)$ can be uniquely expanded into

$$f(i, r; j, s) = \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} c_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} \times g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}(i, r; j, s). \quad (45)$$

Take the ZT of both sides of (45). Applying Theorem 7, we can write

$$(\mathscr{Z}_L f)(i, j; \rho, \sigma) = (\mathscr{Z}_L g_{m'',n'',k'',l''})(i, j; \rho, \sigma) \times \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} c_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} \times \exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \exp\left(-2\pi\iota \frac{m'\rho + n'\sigma}{L}\right) \times \exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right). \quad (46)$$

The inverse $(M \times L) \times (M \times L)$ four-dimensional FT of the quotient function

$$\frac{\mathscr{Z}_L f(i, j; \rho, \sigma)}{\mathscr{Z}_L g_{m'',n'',k'',l''}(i, j; \rho, \sigma)}, \quad 0 \le i, j < M, 0 \le \rho, \sigma < L, \quad (47)$$

computes the Gabor coefficients of $f$ relative to the shifted basis

$$g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}, \quad 0 \le m', n' < L, 0 \le k', l' < M.$$

We now derive a formula relating the shifted basis to the coarse basis. The Gabor expansion of the shifted basis with respect to the coarse basis can be written as

$$g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}(i, r; j, s) = \sum_{\mu=0}^{L-1} \sum_{\nu=0}^{L-1} \sum_{\kappa=0}^{M-1} \sum_{\lambda=0}^{M-1} d_{\mu M,\nu M,\kappa L,\lambda L} \times g_{\mu M,\nu M,\kappa L,\lambda L}(i, r; j, s), \quad (48)$$

where $\{d_{\mu M,\nu M,\kappa L,\lambda L}\}$ are the expansion coefficients. Taking the ZT of both sides and applying Theorem 7, we have

$$\mathscr{Z}_L g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}(i, j; \rho, \sigma) = \mathscr{Z}_L g(i, j; \rho, \sigma) \times \sum_{\mu=0}^{L-1} \sum_{\nu=0}^{L-1} \sum_{\kappa=0}^{M-1} \sum_{\lambda=0}^{M-1} d_{\mu M,\nu M,\kappa L,\lambda L} \times \exp\left(-2\pi\iota \frac{\mu\rho + \nu\sigma}{L}\right) \exp\left(-2\pi\iota \frac{i\kappa + j\lambda}{M}\right). \quad (49)$$

The inverse $(M \times L) \times (M \times L)$ four-dimensional FT of the quotient function

$$\frac{\mathfrak{Z}_L g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}(i, j; \rho, \sigma)}{\mathfrak{Z}_L g(i, j; \rho, \sigma)}, \quad 0 \le i, j < M, 0 \le \rho, \sigma < L, \tag{50}$$

computes the Gabor coefficients of an image $f$ relative to the coarse basis

$$g_{\mu M, \nu M, \kappa L, \lambda L}, \quad 0 \le \mu, \nu < L, 0 \le \kappa, \lambda < M.$$

By applying Theorem 7 to (50) we can rewrite it as

$$\exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \frac{(\mathfrak{Z}_L g_{m'',n'',k'',l''})(i, j; \rho, \sigma)}{\mathfrak{Z}_L g(i, j; \rho, \sigma)} \times \exp\left(-2\pi\iota \frac{m'\rho + n'\sigma}{L}\right) \exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right), \quad 0 \le i, j < M, 0 \le \rho, \sigma < L, \tag{51}$$

The quotient function (51) can be rewritten as

$$\exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) H(i, j; \rho, \sigma) \times \exp\left(-2\pi\iota \frac{m'\rho + n'\sigma}{L}\right) \exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right), \quad 0 \le i, j < M, 0 \le \rho, \sigma < L, \tag{52}$$

where

$$H(i, j; \rho, \sigma) = \frac{(\mathfrak{Z}_L g_{m'',n'',k'',l''})(i, j; \rho, \sigma)}{\mathfrak{Z}_L g(i, j; \rho, \sigma)}, \quad 0 \le i, j < M, 0 \le \rho, \sigma < L. \tag{53}$$

Denote the $(M \times L) \times (M \times L)$ four-dimensional Fourier transform of $H$ by $\hat{H}$. Straightforward computation shows that the FT of (53) is

$$d_{\mu,\nu,\kappa,\lambda} = \exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \times \hat{H}(k' - \mu, l' - \nu, m' - \kappa, n' - \lambda), \tag{54}$$

proving the next result.

THEOREM 8. *For $0 \le m'' < M$, $0 \le n'' < L$,*

$$g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L}(i, r; j, s) = \exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \times \sum_{\mu=0}^{L-1} \sum_{\nu=0}^{L-1} \sum_{\kappa=0}^{M-1} \sum_{\lambda=0}^{M-1} \hat{H}(k' - \mu, l' - \nu, m' - \kappa, n' - \lambda) \times g_{\mu M,\nu M,\kappa L,\lambda L}(i, r; j, s), \tag{55}$$

*where $H$ is given in* (53) *and depends on $m''$, $n''$, $k''$, $l''$.*

This theorem relates the shifted basis to a coarse basis and affords the refinement of a given grid for the purpose of obtaining any desirable increasing in resolution. Related problems of $S/N$ are dealt with elsewhere [4]. The following corollary is an immediate consequence of Theorem 8:

COROLLARY 3. *For $0 \le m''$, $n'' < M$, $0 \le k''$, $l'' < L$,*

$$\langle f, g_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} \rangle = \exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \times \sum_{\mu=0}^{L-1} \sum_{\nu=0}^{L-1} \sum_{\kappa=0}^{M-1} \sum_{\lambda=0}^{M-1} \hat{H}(k' - \mu, l' - \nu, m' - \kappa, n' - \lambda) \times \langle f, g_{\mu M,\nu M,\kappa L,\lambda L} \rangle, \tag{56}$$

*where $H$ is given in* (53).

We now relate the Gabor coefficients to those relating the shifted basis to the coarse basis. From (36) and (46) we get

$$\frac{\mathfrak{Z}_L g(i, j; \rho, \sigma)}{\mathfrak{Z}_L g_{m'',n'',k'',l''}(i, j; \rho, \sigma)} \times \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} c_{m'M,n'M,k'L,l'L} \times \exp\left(-2\pi\iota \left(\frac{(m'\rho}{L} + \frac{n'\sigma}{L}\right)\right) \exp\left(-2\pi\iota \left(\frac{k'i}{M} + \frac{l'j}{M}\right)\right)$$
$$= \sum_{m'=0}^{L-1} \sum_{n'=0}^{L-1} \sum_{k'=0}^{M-1} \sum_{l'=0}^{M-1} c_{m''+m'M,n''+n'M,k''+k'L,l''+l'L} \times \exp\left(2\pi\iota \frac{m'k'' + n'l''}{L}\right) \exp\left(-2\pi\iota \frac{m'\rho + n'\sigma}{L}\right) \times \exp\left(-2\pi\iota \frac{ik' + jl'}{M}\right). \tag{57}$$

COROLLARY 4.

$$c_{m''+\mu M,n''+\nu M,k''+\kappa L,l''+\lambda L} = \exp\left(-2\pi\iota \frac{\mu k'' + \nu l''}{L}\right) \hat{H}^{-1} ** c_{\mu M,\nu M,\kappa L,\lambda L}, \tag{58}$$

*where*

$$H^{-1}(i, j; \rho, \sigma) = \frac{\mathfrak{Z}_L g(i, j; \rho, \sigma)}{(\mathfrak{Z}_L g_{m'',n'',k'',l''})(i, j; \rho, \sigma)}, \quad 0 \le i, j < M, 0 \le \rho, \sigma < L, \tag{59}$$

Note that in both (57) and (58), the series have the form of a four-dimensional cyclic convolution. The presented formulae enable the computation of the Gabor expansion coefficients with progressively increased resolution.

## 7. IMPLEMENTATION AND CONCLUSION

It was previously argued that images [11] and a variety of one-dimensional signals [13] are nonstationary, and that Gabor-type expansion is in some sense an optimal technique for the representation of such signals. The FZT is in particular suitable for computation of the Gabor expansion coefficients [13]. Since the ZT (and FZT) is information preserving, the image can be fully recovered from its ZT. The only condition required of a Gabor-type (windowed) wavelet function, for its implementation in a decomposition and representation of an image by means of a stable computation of the FZT, is that the ZT of the function not vanish over the fundamental four-dimensional cube. This condition is satisfied by many Gabor-type wavelet functions which can be synthesized for, and are of interest in, image representation. In the case of a Gaussian, which is of special interest because of its smoothness and optimal quality of position–frequency localization [11, 5], the ZT contains a zero plane. The existence of such a zero plane, which is a direct consequence of the "Zero Theorem" [13], is in fact the reason for the instability encountered in the implementation of the Gabor scheme using the biorthogonal functions [11]. Using the Gabor elementary function (Gaussian window), or any other function which has a continuous ZT and therefore does not satisfy the above stated condition, one can manipulate the function so as to shift the zero of its ZT outside of the fundamental four-dimensional cube. Such a regularization approach was adopted by Auslander *et al.* in their application of the one-dimensional discrete FZT to time–frequency analysis of nonstationary signals [13]. An alternative approach is to increase the resolution by overlaying grids as presented in this paper, and to shift the axis of symmetry by a subpixel. The absolute minimum value of the ZT of the Gaussian then becomes a function of the subpixel distance by which the Gaussian has been shifted. It has been shown that there is an optimum value of shift which maximizes the minimum value of the ZT of the shifted Gaussian [13, 24].

The example depicted in Fig. 1 illustrates computational results obtained by implementation of the FZT technique, applied in Gabor decomposition and reconstruction from a partial set of Gabor coefficients using stabilization by subpixel shifting of the Gaussian. The sequence (from left to right) of the top row of images corresponds to 600, 1000, 3000, and 9000 coefficients. For comparison, the results of reconstruction from the same number of coefficients using the biorthogonal functions are shown (bottom row) [11]. This comparison demonstrates that in order to obtain results of a perceptually similar quality, the number of components (coefficients) required with the FZT is about 1/10 of that required while using the biorthogonal technique.

Another important application of the proposed technique is in processing of images with progressive resolution. Theorem 8 indicates how the grid can be iteratively refined by the shifting algorithm for the purpose of progressively increased resolution. The filter defined by (58) can be constructed (realized) as a programmable filter and updated at each step of the iteration by the corresponding parameters. Further, the proposed approach offers new insight into the important issue of subsampling without aliasing.

Thus, the FZT offers a powerful technique for image representation and analysis in the combined position–

FIG. 1. Gabor decomposition and reconstruction from a partial set of coefficients. Top row: using Zak transform. Bottom row: using biorthogonal dual (gamma) functions. The sequence from left to right corresponds to 600, 1000, 3000, and 9000 coefficients.

frequency space. Being a DFT-based technique, it also lends itself to interesting VLSI architectural designs which may afford on-line real-time implementations.

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IZIDOR GERTNER is a research professor at the CAIP Center, Rutgers University. He is active with the groups working in the areas of vision, parallel computers, and visualization. Professor Gertner has a deep and broad experience with algorithms for fast computation of transforms. He is in particular well known for the development of parallel algorithms that are instrumental in the analysis and synthesis of images and other types of signals. He has extensive experience in implementation of fast algorithms on parallel computers and has served as a consultant in this area of his expertise to some of the leading companies in this field including IBM, AT&T, and DEC. Another area of his expertise is related to efficient image representation for the purpose of target detection and tracking. Profession Gertner was born in Lithuania and received there degrees in both mathematics and electrical engineering. He earned his Ph.D. degree from the Technion–Israel institute of Technology and was on its faculty for several years. He was also on the faculty of CUNY where he served as a research professor in the Center for Large Scale Computation.